J. Fluid Mech. (1974), *vol.* 62, *part* 4, *pp.* 775–791
Printed in Great Britain

Edge waves: a long-wave theory for oceans of finite depth

By R. GRIMSHAW
Department of Mathematics, University of Melbourne, Victoria 3052, Australia

(Received 25 June 1973)

Edge waves travelling along a straight coastline are examined in the case when the water depth approaches a constant value at large distances from the coast. Only the fundamental mode is examined in the limit as the ratio of the water depth at infinity to the edge-wave wavelength tends to zero. Two comparison theorems are used to obtain upper and lower bounds for the dispersion relation. A long-wave approximation procedure is used to obtain the leading terms in the dispersion relation for a wide class of bottom topographies. The results obtained are compared with an exact result for the case when the bottom topography is a rectangular step.

1. Introduction

It was observed by Stokes (1846; see Wehausen 1960, §18) that edge waves can propagate along a straight coast when the water depth has a constant slope. These waves have a frequency $(gl\sin\beta)^{\frac{1}{2}}$, where l is the wavenumber and β is the slope of the bottom; the amplitude decreases exponentially with distance from the coast, and the energy is finite and effectively confined to within a wavelength of the coast. Ursell (1952) showed that the Stokes edge wave is the first in a sequence of edge waves, where the nth mode has a frequency $(gl\sin[2n+1]\beta)^{\frac{1}{2}}$, and $(2n+1)\beta \leqslant \frac{1}{2}\pi$. Thus for a given slope, there are only a finite number of modes, the number increasing as the slope decreases.

There would seem to be no other exact edge-wave solutions known ('exact' here being within the context of the inviscid irrotational theory of linearized water waves). Eckart (1951) used the shallow-water approximation to examine edge waves on a beach of constant slope, and found an infinite set of modes with frequencies $(gl[2n+1]\tan\beta)^{\frac{1}{2}}$, $n = 1, 2, 3, \ldots$; comparison with Ursell's results (1952) indicates agreement for sufficiently small values of $(2n+1)\beta$; also the exact theory forecasts only a finite number of modes, while the shallow-water approximation predicts an infinite number. Ball (1967) and Longuet-Higgins (1967) also used the shallow-water approximation; both found an infinite set of edge-wave modes; Ball's results were for an exponential depth profile and Longuet-Higgins's results were for a rectangular step. Both authors also considered the effect of rotation. Shen, Meyer & Keller (1968) used a different approximation procedure based on the smallness of the bottom slope; their procedure used the ratio of the edge-wave wavelength to the trapping distance from the coast as a large parameter, and is therefore expected to be most useful for the higher modes. For a recent survey of this procedure and the shallow-water approximation, see Meyer (1971).

In this paper we shall consider edge waves over water whose depth approaches a constant value at large distances from the coast. Although it has been conjectured that, for this geometry, there is an infinite set of edge-wave modes, we shall concentrate attention on the fundamental mode for which the frequency and wavenumber simultaneously tend to zero. In § 2 the problem is formulated. In § 3 two comparison theorems are established and used to obtain bounds on the dispersion relation connecting the frequency and wavenumber. For the lower bound an extra assumption (3.15) is made concerning the depth profile. In § 4 a long-wave approximation procedure is presented, and is based on the ratio of the depth at infinity (i.e. a large distance from the coast) to the edge-wave wavelength being a small number. In this section a different assumption (4.1), or (4.33), is made concerning the depth profile. In § 5 the shallow-water approximation procedure is examined in relation to the long-wave approximation used in § 4. In § 6 the case when the depth profile is a rectangular step is examined through an exact integral equation. The dispersion relation for the fundamental edge wave is obtained and the results of §§ 4 and 5 confirmed for this special case.

2. Formulation

We shall assume that the depth contours are parallel to the straight coastline, and that far from the coast the depth tends to a constant value h_0. The equations will be presented in dimensionless form, using h_0 as the unit of length, $(gh_0)^{\frac{1}{2}}$ as the unit of velocity and $(h_0/g)^{\frac{1}{2}}$ as the unit of time. The x axis is directed away from the coast, the z axis is coincident with the coastline and the y axis is vertical. The fluid, in the equilibrium state, occupies the region $\mathscr{S}$, which is bounded by the free surface $\mathscr{F}$, $y = 0$, and the bottom $\mathscr{D}$, $y = -h(x)$, where $h(0) = 0$, $h(x) > 0$ and $h \to 1$ as $x \to \infty$ (see figure 1).

It will be assumed that the fluid is inviscid, incompressible and of constant density, and the discussion will be confined to the linear theory of irrotational surface waves, neglecting surface tension. Then the velocity potential $\phi^*(x, y, z, t)$ satisfies Laplace's equation

$$\frac{\partial^2\phi^*}{\partial x^2}+\frac{\partial^2\phi^*}{\partial y^2}+\frac{\partial^2\phi^*}{\partial z^2} = 0 \quad \text{in} \quad \mathscr{S}, \tag{2.1}$$

and the boundary conditions

$$\frac{\partial^2\phi^*}{\partial t^2}+\frac{\partial\phi^*}{\partial y} = 0 \quad \text{on} \quad \mathscr{F} \tag{2.2}$$

and

$$\partial\phi^*/\partial n = 0 \quad \text{on} \quad \mathscr{D}. \tag{2.3}$$

We are interested in edge waves travelling parallel to the coast and we therefore seek solutions for which

$$\phi^* = \phi(x, y)\cos(lz - \sigma t). \tag{2.4}$$

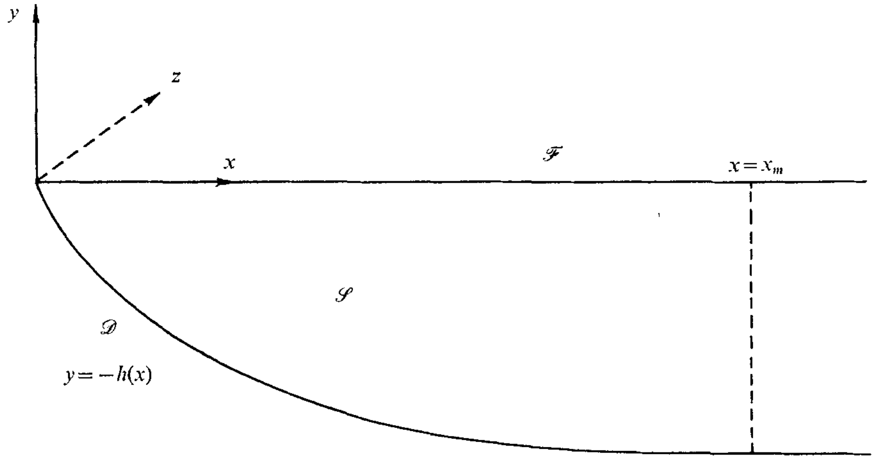

FIGURE 1. Notation.

ϕ then satisfies the equation

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = l^2\phi \quad \text{in} \quad \mathscr{S}, \tag{2.5}$$

and the boundary conditions

$$\partial\phi/\partial y = \sigma^2\phi \quad \text{on} \quad \mathscr{F} \tag{2.6}$$

and

$$\partial\phi/\partial n = 0 \quad \text{on} \quad \mathscr{D}. \tag{2.7}$$

Edge waves are characterized by the extra condition

$$\phi, |\nabla\phi| \to 0 \quad \text{as} \quad x \to \infty. \tag{2.8}$$

In addition, it will be assumed that ϕ and $\nabla\phi$ are continuous throughout $\mathscr{S}$, $\mathscr{F}$ and $\mathscr{D}$, except possibly for integrable singularities at $x = y = 0$ (and possibly at other 'corners' on $\mathscr{D}$); these singularities are sufficiently weak to ensure that ϕ and $\nabla\phi$ are integrable, and square-integrable throughout $\mathscr{S}$.

The existence of a non-trivial solution to (2.5)–(2.8) will be possible only if σ, the frequency, is related to l, the wavenumber, by a *dispersion relation*. Green's theorem in a plane, for a region $\mathscr{A}$ with boundary $\partial\mathscr{A}$, states that

$$\iint_{\mathscr{A}} \nabla.\mathbf{v}\,dx\,dy = \int_{\partial\mathscr{A}} \mathbf{v}.\mathbf{n}\,ds, \tag{2.9}$$

where $\mathbf{v}$ is a two-dimensional vector field and $\mathbf{n}$ is the outward normal to $\partial\mathscr{A}$. Application of this to $\nabla\phi$ over the region $\mathscr{S}$ yields the *compatibility relation*

$$\iint_{\mathscr{S}} \phi\,dx\,dy = c^2\int_{\mathscr{F}} \phi\,dx, \tag{2.10}$$

where

$$c = \sigma/l \text{ is the } \textit{phase velocity}. \tag{2.11}$$

It may be conjectured (cf. Meyer 1971) that there is an infinite sequence of edge-wave modes, and that the fundamental mode is that for which $\sigma \to 0$ as $l \to 0$. In this paper the existence of this fundamental mode will be assumed; given this, our aim is to determine the dispersion relation in the limit $l \to 0$.

A further application of Green's theorem to $\phi\nabla\phi$ shows that

$$\iint_{\mathscr{S}} \{|\nabla\phi|^2 + l^2|\phi|^2\}\,dx\,dy = \sigma^2 \int_{\mathscr{F}} \phi^2\,dx. \tag{2.12}$$

The left-hand side of this equation is (one quarter of) the kinetic energy of the edge wave, and the right-hand side is (one quarter of) the potential energy. It will be assumed that the condition (2.8) is sufficient to ensure that the integrals in (2.12) (and also (2.9)) converge, as edge waves are characterized by having *finite energy* (Meyer 1971).

3. Comparison theorems

Two theorems will be established, both of which depend on the vector identity

$$|\nabla\phi|^2 + l^2\phi^2 \equiv \psi^2|\nabla v|^2 + v^2\psi(l^2\psi - \nabla^2\psi) + \nabla\,.\,(v^2\psi\nabla\psi), \tag{3.1}$$

where

$$\phi \equiv \psi v. \tag{3.2}$$

Let $\mathscr{S}_m$ (and similarly $\mathscr{D}_m$ and $\mathscr{F}_m$) be the restriction of $\mathscr{S}$ to $x \leqslant x_m$, where $\{x_m\}$ is a monotonic sequence of positive numbers such that $x_m \to \infty$ as $m \to \infty$ (cf. figure 1). Then, from Green's theorem (2.9),

$$\iint_{\mathscr{S}_m} (|\nabla\phi|^2 + l^2\phi^2)\,dx\,dy - \sigma^2\int_{\mathscr{F}_m} \phi^2\,dx = \iint_{\mathscr{S}_m} [\psi^2|\nabla v|^2 + v^2\psi(l^2\psi - \nabla^2\psi)]\,dx\,dy$$
$$+ \int_{x=x_m} \psi\psi_x v^2\,dy + \int_{\mathscr{D}_m} v^2\psi\frac{\partial\psi}{\partial n}\,ds + \int_{\mathscr{F}_m} v^2\psi(\psi_y - \sigma^2\psi)\,dx. \tag{3.3}$$

THEOREM 1. Let σ be the frequency of an edge wave of wavenumber l, and let ψ (the comparison function) satisfy the conditions

$$\psi > 0 \quad \text{in} \quad \mathscr{S}, \mathscr{F}, \mathscr{D}, \tag{3.4}$$

$$\nabla^2\psi - l^2\psi \leqslant 0 \quad \text{in} \quad \mathscr{S}, \tag{3.5}$$

$$\partial\psi/\partial n \geqslant 0 \quad \text{on} \quad \mathscr{D}, \tag{3.6}$$

$$\psi_y - \sigma_1^2\psi \geqslant 0 \quad \text{on} \quad \mathscr{F}, \tag{3.7}$$

$$\psi_x/\psi \quad \text{bounded as} \quad x \to \infty. \tag{3.8}$$

Then

$$\sigma^2 \geqslant \sigma_1^2. \tag{3.9}$$

Proof. Let ϕ be an edge wave in (3.2); then in (3.3), using (3.4)–(3.7), it follows that

$$\iint_{\mathscr{S}_m} (|\nabla\phi|^2 + l^2\phi^2)\,dx\,dy - \sigma^2\int_{\mathscr{F}_m} \phi^2\,dx \geqslant \int_{x=x_m} \frac{\psi_x}{\psi}\phi^2\,dy + (\sigma_1^2 - \sigma^2)\int_{\mathscr{F}_m} v^2\psi^2\,dx. \tag{3.10}$$

As $m \to \infty$, the left-hand side of (3.10) tends to zero (using (2.12)); from (2.8) and (3.8) the first term on the right-hand side also tends to zero as $m \to \infty$. The result (3.9) follows on letting $m \to \infty$ in (3.10).

This comparison theorem is related to similar theorems which have been established for eigenvalue problems associated with the Laplacian operator in bounded domains (cf. Hersch & Payne (1968), and the references there). The result can also be established in a vector form. Let $\mathbf{p}$ be a vector such that

$$\left.\begin{aligned} \nabla . \mathbf{p} + |\mathbf{p}|^2 \leqslant l^2 \quad \text{in} \quad \mathscr{S}, \quad \mathbf{p}.\mathbf{n} \geqslant 0 \quad \text{on} \quad \mathscr{D}, \quad \mathbf{p}.\mathbf{n} \geqslant \sigma_1^2 \quad \text{on} \quad \mathscr{F}, \\ \mathbf{p}.\mathbf{i} \quad \text{is bounded as} \quad x \to \infty. \end{aligned}\right\} \tag{3.11}$$

Here $\mathbf{i}$ is a unit vector in the x direction. Then it may be shown that (3.9) is obtained as before; the proof is similar to that of theorem 1 (cf. Hersch & Payne 1968). Theorem 1 is recovered on letting $\mathbf{p} = \nabla\psi/\psi$.

THEOREM 2. Let χ be an edge wave of frequency σ and wavenumber l (i.e. a solution of (2.5)–(2.8)) which satisfies the conditions

$$\chi > 0 \quad \text{in} \quad \mathscr{S}, \mathscr{F}, \mathscr{D}, \quad \chi_x/\chi \quad \text{bounded as} \quad x \to \infty. \tag{3.12}$$

Let ϕ be any function which satisfies the condition

$$\phi \to 0 \quad \text{as} \quad x \to \infty. \tag{3.13}$$

Then

$$\sigma^2 \int_{\mathscr{F}} \phi^2\, dx \leqslant \iint_{\mathscr{S}} (|\nabla\phi|^2 + l^2\phi^2)\, dx\, dy. \tag{3.14}$$

Proof. Let $\psi \equiv \chi$ in (3.2); then since $\chi > 0$, any function ϕ may be expressed in the form (3.2), and so (3.3) becomes

$$\iint_{\mathscr{S}_m} (|\nabla\phi|^2 + l^2\phi^2)\, dx\, dy - \sigma^2 \int_{\mathscr{F}_m} \phi^2\, dx \geqslant \int_{x=x_m} \frac{\chi_x}{\chi} \phi^2\, dy.$$

The right-hand side vanishes as $m \to \infty$ on using (3.12) and (3.13). The result (3.14) follows on letting $m \to \infty$. Although there is no proof available that there exists a positive edge wave, the results of §4 make it plausible that the fundamental edge-wave mode satisfies the conditions (3.12). Theorem 2 is a form of Rayleigh's principle.

Theorems 1 and 2 together provide upper and lower bounds for the dispersion relation. First we shall apply theorem 1 for the class of depth profiles satisfying the condition

$$\left.\begin{aligned} h' \geqslant M(1-h) \geqslant 0, \\ M > 0. \end{aligned}\right\} \tag{3.15}$$

where

This condition implies that $1 - h \geqslant e^{-Mx}$, and in particular, that $h'(0) \geqslant M$. For a comparison function, choose

$$\psi = \cosh k_1(1+y) \exp(-\gamma_1 x), \tag{3.16}$$

where

$$\gamma_1^2 = l^2 - k_1^2 > 0 \tag{3.17}$$

and

$$\sigma_1^2 = k_1 \tanh k_1. \tag{3.18}$$

Then ψ satisfies (3.4), (3.8), (3.5) and (3.7), the latter two with equality. Also

$$\left.\frac{\partial\psi}{\partial n}\right|_{\mathscr{D}} = \frac{-h'}{(1+h'^2)^{\frac{1}{2}}}\psi_x - \frac{\psi_y}{(1+h'^2)^{\frac{1}{2}}}, \tag{3.19}$$

and so the remaining condition on ψ, inequality (3.6), will be satisfied if

$$\gamma_1 h' \geqslant k_1 \tanh k_1 (1-h). \tag{3.20}$$

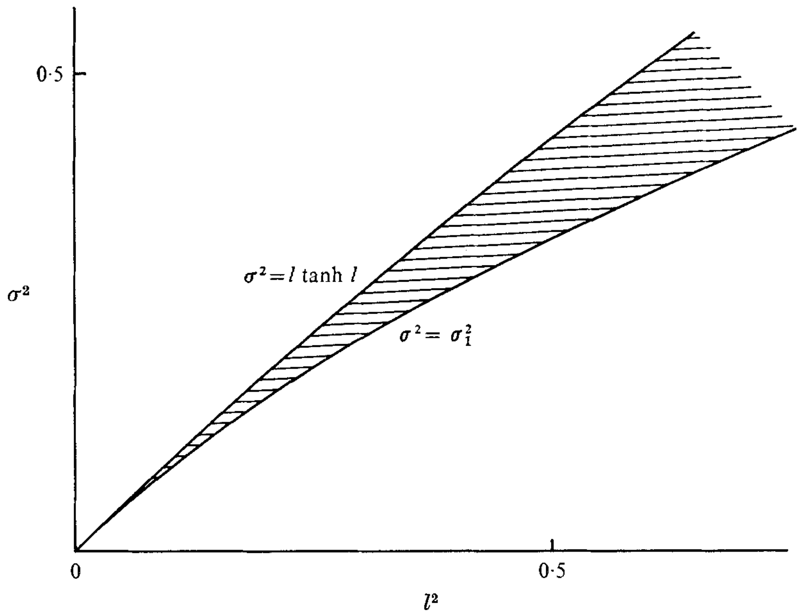

FIGURE 2. Graph of σ^2 against ι^2. The shaded area indicates the region in which the dispersion relation may be satisfied. The curve $\sigma^2 = \sigma_1^2$ is shown for $M = 1$.

But $\tanh \xi \leqslant \xi$ for all non-negative ξ, and hence (3.20) will be satisfied if we choose k_1 so that

$$k_1^2 = \gamma_1 M, \tag{3.21}$$

where M is defined by (3.15). Equations (3.17), (3.18) and (3.21) determine σ_1 as a function of l and theorem 1 now implies that $\sigma^2 \geqslant \sigma_1^2$, for depth profiles $\mathscr{D}$ satisfying (3.15). For small values of l^2, equation (3.21) implies that

$$\sigma_1^2 = l^2 - l^4\left(\frac{1}{3}+\frac{1}{M^2}\right) + O\left\{l^6\left(\frac{M^2+1}{M^2}\right)\right\}. \tag{3.22}$$

A graph of σ_1^2 versus l^2 is shown in figure 2 (for $M = 1$). It is possible to relax the condition (3.15) on the depth profile $\mathscr{D}$ by choosing a different comparison function, but the results are not as tidy as (3.21) and will therefore not be presented here.

The results of §4 show that, if the depth profile $\mathscr{D}$ eventually becomes flat (cf. (4.1): $h = 1$ for all $x > x_0$ say), then

$$c^2 \leqslant \tanh l/l, \quad \text{or} \quad \sigma^2 \leqslant l \tanh l. \tag{3.23}$$

This inequality may also be established from theorem 2, without using the assumption (4.1), provided that it is assumed that there exists an edge wave satisfying (3.12). The proof involves putting

$$\phi = \cosh l(1+y)\exp(-\epsilon x) \tag{3.24}$$

in (3.14), and hence showing that

$$\sigma^2 \leqslant l \tanh l + 2\epsilon \int_{\mathscr{D}} \phi \frac{\partial \phi}{\partial n}\, ds \Big/ \cosh^2 l. \tag{3.25}$$

But
$$\left| \int_{\mathscr{D}} \phi \frac{\partial \phi}{\partial n} ds \Big/ \cosh^2 l \right| \leqslant Q, \tag{3.26}$$

where Q is a constant depending on l, and h, but independent of ϵ. The result (3.23) is now obtained by letting $\epsilon \to 0$.

We have now obtained both upper and lower bounds on the dispersion relation (assuming, in particular, that (3.15) holds), and these are displayed in figure 2, where the shaded region indicates the allowed values of σ^2, viz.

$$\sigma_1^2 \leqslant \sigma^2 \leqslant l \tanh l. \tag{3.27}$$

In particular, (3.22) implies that

$$c^2 \to 1 \quad \text{as} \quad l^2 \to 0. \tag{3.28}$$

4. Long-wave approximation

In this section it will be assumed that

$$h = 1 \quad \text{for} \quad x > x_0, \tag{4.1}$$

where x_0 is $O(1)$ (as $l \to 0$). The class of depth profiles defined by (4.1) is different from that defined by (3.15); nevertheless, since many depth profiles satisfy both (3.15) and (4.1), we shall use the results of § 3 to draw conclusions about the nature of the dispersion relation when (4.1) is assumed.

Let k and k_m $(m = 1, 2, 3)$ be the positive solutions of the equations

$$\sigma^2 = k \tanh k, \tag{4.2}$$

$$\sigma^2 = -k_m \tan k_m \quad (m = 1, 2, 3, \ldots). \tag{4.3}$$

The functions in the set $\{\cosh k(1+y),\ \cos k_m(1+y)\}$ then form a complete orthogonal set over the interval $-1 \leqslant y \leqslant 0$ (see Wehausen 1960, § 16) and satisfy the boundary conditions (2.6) (at $y = 0$) and (2.7) (at $y = -1$). Hence

$$\phi = a \cosh k(1+y) \exp(-\gamma x) + \sum_1^\infty a_m \cos k_m (1+y) \exp(-\gamma_m x) \quad (x > x_0), \tag{4.4}$$

where
$$\gamma = (l^2 - k^2)^{\frac{1}{2}} > 0, \quad \gamma_m = (l^2 + k_m^2)^{\frac{1}{2}} > 0. \qquad (4.5), (4.6)$$

A necessary condition for edge waves is $\gamma^2 > 0$; thus $l^2 > k^2$, and then (4.2) implies that $\sigma^2 < l \tanh l$. The coefficient a will be regarded as an arbitrary positive constant; the coefficients a_m depend linearly on a. It will emerge in the analysis to follow that the a_m are $O(al^2)$ as $l \to 0$ (see (4.26)), and hence that (4.4) defines an edge wave for which $\phi > 0$ everywhere (at least as $l \to 0$).

The results (3.27) and (3.28) suggest that the dispersion relation may be expressed in the form

$$\left.\begin{aligned} c^2 &= 1 + \beta l^2 + O(l^4), \\ \text{or} \qquad \sigma^2 &= l^2 + \beta l^4 + O(l^6). \end{aligned}\right\} \tag{4.7}$$

Since, from (4.2) and (4.3),

$$k^2 = \sigma^2 + \tfrac{1}{3}\sigma^4 + O(\sigma^6), \tag{4.8}$$

$$k_m = \pi m - \sigma^2/\pi m + O(\sigma^4), \tag{4.9}$$

where the error term in (4.9) is independent of m, it follows that

$$k^2 = l^2 + (\beta + \tfrac{1}{3}) l^4 + O(l^6), \tag{4.10}$$

$$k_m = \pi m - l^2/\pi m + O(l^4). \tag{4.11}$$

Then, from (4.5) and (4.6), we have

$$\gamma^2 = -(\beta + \tfrac{1}{3}) l^4 + O(l^6), \tag{4.12}$$

$$\gamma_m = \pi m - l^2/2\pi m + O(l^4). \tag{4.13}$$

A necessary condition for the existence of edge waves is

$$-\beta > \tfrac{1}{3}. \tag{4.14}$$

One consequence of (4.12) is that γ is $O(l^2)$ as $l \to 0$, and that γ_m is bounded away from zero for small values of l.

We now let

$$\phi = w + v, \tag{4.15}$$

where
$$w = a \cosh k(1+y) \exp(-\gamma x) \tag{4.16}$$

and
$$v = \sum_1^\infty a_m \cos k_m (1+y) \exp(-\gamma_m x) \quad \text{for} \quad x > x_0. \tag{4.17}$$

w may be regarded as a known function, and v satisfies the equation

$$\nabla^2 v = l^2 v \quad \text{in} \quad \mathscr{S}, \tag{4.18}$$

and the boundary conditions

$$\partial v/\partial y = \sigma^2 v \quad \text{on} \quad \mathscr{F}, \tag{4.19}$$

$$\partial v/\partial n = -\partial w/\partial n \quad \text{on} \quad \mathscr{D}. \tag{4.20}$$

From (4.13) it follows that $v \to 0$ (exponentially) as $x \to \infty$, independently of l as $l \to 0$. Also,

$$-\frac{\partial w}{\partial n}\bigg|_{\mathscr{D}} = \left\{\frac{-h'}{(1+h'^2)^{\frac{1}{2}}} a\gamma \cosh k(1-h) + \frac{ak \sinh k(1-h)}{(1+h'^2)^{\frac{1}{2}}}\right\} \exp(-\gamma x). \tag{4.21}$$

This is zero for $x > x_0$, and for $x < x_0$,

$$-\frac{\partial w}{\partial n}\bigg|_{\mathscr{D}} = al^2 \left\{-(-\beta - \tfrac{1}{3})^{\frac{1}{2}} \frac{h'}{(1+h'^2)^{\frac{1}{2}}} + \frac{(1-h)}{(1+h'^2)^{\frac{1}{2}}}\right\} + O(al^4). \tag{4.22}$$

Since x_0 is $O(1)$, this expansion is uniform in x.

Equation (4.22) implies that v may be expanded in the form

$$v = a(l^2 v_1 + l^4 v_2 + \ldots). \tag{4.23}$$

Substituting into (4.18), (4.19) and (4.22) we have

$$\left.\begin{aligned} &\nabla^2 v_1 = 0 \quad \text{in} \quad \mathscr{S}, \qquad \partial v_1/\partial y = 0 \quad \text{on} \quad \mathscr{F}, \\ &\frac{\partial v_1}{\partial n} = -(-\beta - \tfrac{1}{3})^{\frac{1}{2}} \frac{h'}{(1+h'^2)^{\frac{1}{2}}} + \frac{(1-h)}{(1+h'^2)^{\frac{1}{2}}} \quad \text{on} \quad \mathscr{D}. \end{aligned}\right\} \tag{4.24}$$

Since $v \to 0$ as $x \to \infty$, independently of l, it follows that $v_1 \to 0$ as $x \to \infty$ and hence

$$v_1 = \sum_1^\infty a_{m1} \cos \pi m y \exp(-\pi m x) \quad \text{for} \quad x > x_0. \tag{4.25}$$

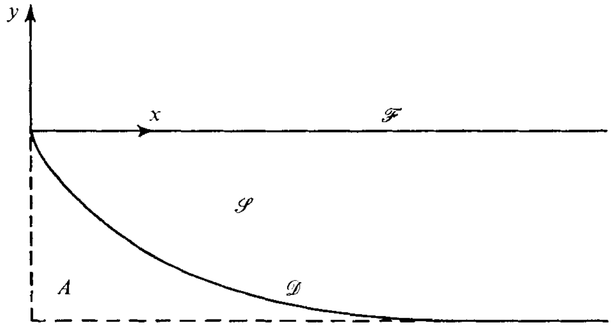

FIGURE 3. Definition of A.

Comparing this with (4.17) it follows that

$$a_m = al^2 a_{m1}(-1)^m + O(al^4). \tag{4.26}$$

Application of Green's theorem (2.9) to ∇v_1 yields the compatibility relation

$$\int_{\mathscr{D}} \frac{\partial v_1}{\partial n} ds = 0, \tag{4.27}$$

or

$$\int_0^\infty (1-h)\,dx = (-\beta - \tfrac{1}{3})^{\frac{1}{2}}. \tag{4.28}$$

Of course, this result could have been obtained by substituting (4.15) into the exact compatibility relation (2.10). In (4.28) the upper limit of integration is really x_0 as $h = 1$ for $x > x_0$. We have thus found

$$-\beta = A^2 + \tfrac{1}{3},$$

where

$$A = \int_0^\infty (1-h)\,dx. \tag{4.29}$$

A is the 'area' under the depth profile (see figure 3). A consequence of (4.28) is that a necessary condition for the existence of an edge wave is $A > 0$.

It has been assumed up till now that the depth profile can be represented by $y = -h(x)$, where h is a single-valued function of x for $x > 0$. This is clearly not necessary and 're-entrant beaches' (see figure 4) are permissible; for such beaches (4.28) is replaced by

$$A - A' = (-\beta - \tfrac{1}{3})^{\frac{1}{2}}, \tag{4.30}$$

where A is defined in figure 4; if S_{x0} is the area occupied by the fluid for $x \leqslant x_0$, then $A - A' = x_0 - S_{x0}$. A necessary condition for the existence of an edge wave is $A > A'$.

We shall now omit the assumption (4.1). Retaining the decomposition (4.15), it follows from the compatibility relation (2.10) (or equivalently, applying Green's theorem to ∇v) that

$$l^2 \iint_{\mathscr{S}} v\,dx\,dy - \sigma^2 \int_{\mathscr{F}} v\,dx = -\int_{\mathscr{D}} \frac{\partial w}{\partial n} ds. \tag{4.31}$$

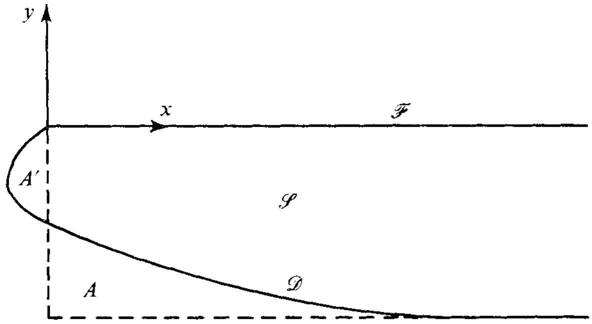

FIGURE 4. Definition of A and A' for a 're-entrant' beach.

Using (4.21) and integrating by parts, we have

$$-\int_{\mathscr{D}} \frac{\partial w}{\partial n} ds = a\left[-\frac{\gamma \sinh k}{k} + l^2 \int_0^\infty \frac{\sinh k(1-h)}{k} dx - \gamma l^2 \int_0^\infty \exp(-\gamma x)\left\{\int_x^\infty \frac{\sinh k(1-h)}{k} dx'\right\} dx\right]. \quad (4.32)$$

Further progress depends on establishing that v is $O(al^2)$, and that $v \to 0$ as $x \to \infty$ (independently of l); then the left-hand side of (4.31) would be $O(al^4)$. Anticipating that γ will again be $O(l^2)$, it follows from (4.21) that $\partial w/\partial n$ is $O(al^2)$ on $\mathscr{D}$; however, this is not sufficient to deduce that v is $O(al^2)$. It seems plausible that v will be $O(al^2)$, uniformly in x, if the 'integrated input'

$$\int_{\mathscr{D}} \left|\frac{\partial w}{\partial n}\right| ds = O(al^2).$$

This will be the case if

$$\int_0^\infty |1-h|\, dx = O(1) \quad \text{as} \quad l \to 0; \quad (4.33)$$

this condition now replaces (4.1). In addition it must be assumed that $h \to 1$ exponentially as $x \to \infty$ with a decay rate which is independent of l; this is to ensure that $v \to 0$ as $x \to \infty$ independently of l. With these assumptions, the left-hand side of (4.31) is $O(al^4)$, and hence

$$\gamma \frac{\sinh k}{k} = l^2 \int_0^\infty \frac{\sinh k(1-h)}{k} dx + O(l^4). \quad (4.34)$$

This result confirms that γ is $O(l^2)$; the result (4.28) is then obtained by equating $O(l^2)$ terms in (4.34).

It may be noted that, for the class of depth profiles satisfying (3.15), $A \leqslant M^{-1}$, so that (4.29) is consistent with (3.27). We conclude this section with two examples. First, suppose that

$$h = 1 - e^{-Mx}; \quad (4.35)$$

(3.15) is satisfied and (4.33) will be satisfied if M is $O(1)$. A simple calculation shows that

$$-\beta = \tfrac{1}{3} + 1/M^2. \quad (4.36)$$

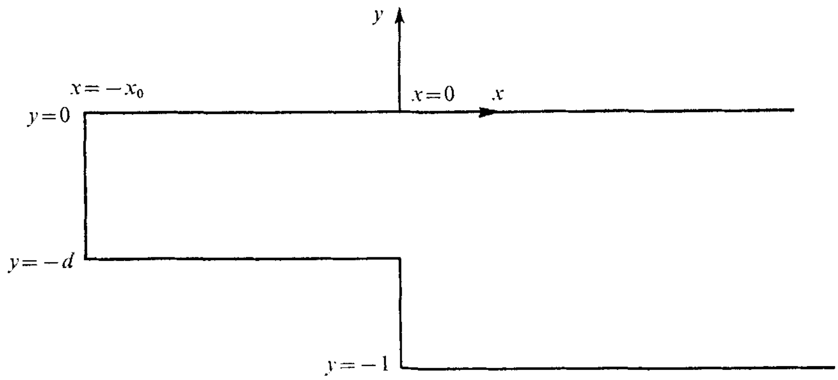

FIGURE 5. Notation for a rectangular step.

Using shallow-water theory (cf. §5) Ball (1967) obtained the result

$$\left.\begin{aligned} \sigma^2+\sigma^4M^{-2} &= l^2, \\ \text{or} \qquad c^2 &= 1-l^2M^{-2}+O(l^4M^{-4}). \end{aligned}\right\} \tag{4.37}$$

Thus the shallow-water theory agrees with the present long-wave approximation when $M \ll 1$, i.e. when the shallow-water dispersive effects dominate over those associated with the '$\frac{1}{3}$' term in (4.29). This latter term has arisen from our use of the correct dispersion relation $\sigma^2 = k\tanh k$ rather than the shallow-water approximation $\sigma^2 \approx k^2$.

Second, let the depth profile D be the rectangular step of depth d and length x_0 (see figure 5). Edge waves exist only if $d < 1$, and then

$$-\beta = \tfrac{1}{3}+x_0^2(1-d)^2. \tag{4.38}$$

In §6, this case will be investigated by an exact integral equation approach and the result (4.38) will be confirmed.

5. Shallow-water approximation

Much of the theoretical work on edge waves has been carried out using the shallow-water approximation (e.g. Eckart 1951; Ball 1967; Longuet-Higgins 1967). This approximation requires not only that $l \ll 1$ but also that the length scale associated with the depth profile $\mathscr{D}$ be comparable with the wavelength $2\pi l^{-1}$. In the present context this means that x_0 in (4.1) is comparable with l^{-1}; the procedure used in §4 therefore fails when lx_0 is only $O(1)$ (or equivalently $l\int|1-h|\,dx$ is only $O(1)$, cf. (4.33)).

An examination of (4.31) and (4.32) shows that in the shallow-water approximation γ is $O(l)$ (and not $O(l^2)$ as in §4, cf. (4.12)), and that the expansion (4.7) for σ^2 is no longer appropriate. We can now identify two regions in $\mathscr{S}$: an inner region where x is $O(1)$ and an outer region where lx is $O(1)$. In the outer region we let

$$X = lx \tag{5.1}$$

and

$$H(X) = h(x). \tag{5.2}$$

X is the appropriate outer variable, in terms of which (2.5)–(2.7) become

$$\phi_{yy} + l^2\phi_{XX} = l^2\phi \quad \text{in} \quad -H < y < 0 \quad (X > 0), \tag{5.3}$$

$$\phi_y = c^2 l^2 \phi \quad \text{on} \quad y = 0 \quad (X > 0), \tag{5.4}$$

$$\phi_y + l^2 \phi_X H'(X) = 0 \quad \text{on} \quad y = -H \quad (X > 0). \tag{5.5}$$

We note that $H'(X)$ equals $l^{-1}h'(x)$. It is a crucial aspect of the shallow-water approximation that $H'(X)$ is $O(1)$ (as $l \to 0$), and so $h'(x)$ must be $O(l')$; this is a significant difference between the long-wave approximation of §4 and the present shallow-water approximation.

We now let

$$\phi = \Phi_0 + l^2\Phi_1 + \ldots. \tag{5.6}$$

Substitution into (5.3)–(5.5) shows that

$$\Phi_0 = F(X) \tag{5.7}$$

and

$$\Phi_{1y} = \begin{cases} (F - F'')y + c^2F & \text{in} \quad -H < y < 0 \quad (X > 0), \quad (5.8) \\ -F'H' & \text{on} \quad y = -H \quad (X > 0). \quad (5.9) \end{cases}$$

It follows that

$$(HF')' + (c^2 - H)F = 0. \tag{5.10}$$

Here a prime denotes a derivative with respect to X. Equation (5.10) is the classical shallow-water approximation in the present context. A significant feature of (5.10) is that $H(X)$ in fact depends on l (cf. equation (5.2)), and hence the solutions of (5.10) will depend on c^2 and on l in a complicated manner. Clearly we must select that solution of (5.10) which decays to zero as $X \to \infty$.

In the shallow-water approximation (5.10) is customarily supplemented by the condition of zero mass flux through the beach, viz.,

$$FH' \to 0 \quad \text{as} \quad X \to 0. \tag{5.11}$$

This is the procedure used by Ball (1967) for the depth profile $\mathscr{D}$ given by (4.35) and Longuet-Higgins (1967) for the rectangular step (figure 5). On integrating (5.10) with respect to X it may be shown that (5.11) implies that

$$c^2 = \int_0^\infty HF\,dX \Big/ \int_0^\infty F\,dX. \tag{5.12}$$

Equation (5.12) may now be recognized as the compatibility relation (2.10) in the present context. Indeed, assuming that $H'(0) \neq 0$, it may be shown that the condition (5.11) implies that $F(0)$ is finite. In this case the matching of the inner expansion (fixed x, $l \to 0$) with the outer expansion leads to the composite expansion

$$\phi = F(X) + O(l^2). \tag{5.13}$$

Thus to lowest order in l, the outer solution contains the inner solution. Substitution of (5.13) into the compatibility relation recovers (5.12).

6. Rectangular step

We now consider the case when the depth profile $\mathscr{D}$ is a rectangular step of depth d and length x_0 (see figure 5). For convenience the axes are relocated so that $x = 0$ is at the edge of the step. This problem can be formulated as an integral equation for the horizontal velocity at the step, using a procedure similar to that employed by Bartholomeusz (1958) and Miles (1967) for the problem of wave transmission over a step.

Let k and k_m $(m = 1, 2, 3, \ldots)$ be the positive solutions of the equations

$$\sigma^2 = k \tanh k = -k_m \tan k_m \tag{6.1}$$

and define

$$\left.\begin{aligned} N^2 &= \int_{-1}^{0} \cosh^2 k(1+y)\,dy = \frac{1}{2}\left\{1 + \frac{\sinh 2k}{2k}\right\}, \\ N_m^2 &= \int_{-1}^{0} \cos^2 k_m(1+y)\,dy = \frac{1}{2}\left\{1 + \frac{\sin 2k_m}{2k_m}\right\}. \end{aligned}\right\} \tag{6.2}$$

Then let

$$\psi(y) = N^{-1}\cosh k(1+y), \quad \psi_m(y) = N_m^{-1}\cos k_m(1+y). \tag{6.3}$$

The functions in the set $\{\psi, \psi_m | m = 1, 2, 3, \ldots\}$ form a complete orthonormal set over the interval $-1 \leqslant y \leqslant 0$ (see Wehausen 1960, § 16) and satisfy the boundary conditions (2.6) (at $y = 0$) and (2.7) (at $y = -1$). Hence

$$\phi = a\psi(y)\exp(-\gamma x) + \sum_1^\infty a_m \psi_m(y)\exp(-\gamma_m x) \quad \text{in} \quad x > 0, \tag{6.4}$$

where

$$\gamma = (l^2 - k^2)^{\frac{1}{2}} > 0, \quad \gamma_m = (l^2 + k_m^2)^{\frac{1}{2}} > 0. \tag{6.5}$$

A similar procedure may now be followed in $x < 0$, where ϕ must satisfy the boundary conditions (2.6) (at $y = 0$), (2.7) (at $y = -d$) and $\phi_x = 0$ at $x = -x_0$. Let k^* and k_m^* $(m = 1, 2, 3, \ldots)$ be the positive solutions of the equations

$$\sigma^2 = k^* \tanh k^* d = -k_m^* \tan k_m^* d, \tag{6.6}$$

and define

$$\left.\begin{aligned} N^{*2} &= \int_{-d}^{0} \cosh^2 k^*(y+d)\,dy = \frac{1}{2}\left\{d + \frac{\sinh 2k^* d}{2k^*}\right\}, \\ N_m^{*2} &= \int_{-d}^{0} \cos^2 k^*(y+d)\,dy = \frac{1}{2}\left\{d + \frac{\sin 2k_m^* d}{2k_m^*}\right\}. \end{aligned}\right\} \tag{6.7}$$

Then let

$$\psi^*(y) = N^{*-1}\cosh k^*(d+y), \quad \psi_m^*(y) = N_m^{*-1}\cos k_m^*(d+y). \tag{6.8}$$

The functions in the set $\{\psi^*, \psi_m^* | m = 1, 2, 3, \ldots\}$ are a complete orthonormal set over $-d \leqslant y \leqslant 0$, and hence

$$\phi = a^* \frac{\cosh \gamma^*(x+x_0)}{\cosh \gamma^* x_0}\psi^*(y) + \sum_1^\infty a_m^* \psi_m^*(y)\frac{\cosh \gamma_m^*(x+x_0)}{\cosh \gamma_m^* x_0} \quad \text{in} \quad -x_0 < x < 0, \tag{6.9}$$

where

$$\gamma^* = (l^2 - k^{*2})^{\frac{1}{2}} = -i(k^{*2} - l^2)^{\frac{1}{2}}, \quad \gamma_m^* = (l^2 + k_m^{*2})^{\frac{1}{2}}. \tag{6.10}$$

The solution (6.9) has been chosen to satisfy the boundary condition at $x = -x_0$ as well as (2.6) and (2.7).

Now define

$$U(y) = \phi_x|_{x=0}. \tag{6.11}$$

Then the boundary condition (2.7) implies that

$$U(y) = 0, \quad -1 < y < -d. \tag{6.12}$$

In addition the matching of the two solutions (6.4) and (6.5) is accomplished by imposing the conditions that ϕ and ϕ_x are continuous at $x = 0$ for $-d < y < 0$. Hence

$$U(y) = -\gamma a\psi(y) - \sum_1^\infty a_m \gamma_m \psi_m(y) \quad (-1 < y < 0), \tag{6.13}$$

$$U(y) = \gamma^* a^* \tanh \gamma^* x_0 \psi^*(y) + \sum_1^\infty \gamma_m^* a_m^* \tanh \gamma_m^* x_0 \psi_m^*(y) \quad (-d < y < 0) \tag{6.14}$$

and
$$a\psi(y) + \sum_1^\infty a_m \psi_m(y) = a^* \psi^*(y) + \sum_1^\infty a_m^* \psi_m^*(y) \quad (-d < y < 0). \tag{6.15}$$

Equations (6.13) and (6.14), together with (6.12), imply that

$$\left.\begin{aligned} -\gamma a &= \int_{-d}^0 U(y)\,\psi(y)\,dy, \\ -\gamma_m a_m &= \int_{-d}^0 U(y)\,\psi_m(y)\,dy, \end{aligned}\right\} \tag{6.16}$$

and
$$\left.\begin{aligned} \gamma^* a^* \tanh \gamma^* x_0 &= \int_{-d}^0 U(y)\,\psi^*(y)\,dy, \\ \gamma_m^* a_m^* \tanh \gamma^* x_0 &= \int_{-d}^0 U(y)\,\psi_m^*(y)\,dy. \end{aligned}\right\} \tag{6.17}$$

Equations (6.16) and (6.17) determine a, a_m, a^* and a_m^* in terms of $U(y)$. Substitution into (6.15) yields

$$\int_{-d}^0 K(y,y')\,U(y')\,dy' = 0 \quad (-d < y < 0), \tag{6.18}$$

where

$$K(y,y') = \gamma^{-1}\psi(y)\,\psi(y') + \sum_1^\infty \gamma_m^{-1} \psi_m(y)\,\psi_m(y') + \gamma^{*-1} \coth \gamma^* x_0 \psi^*(y)\,\psi^*(y') + \sum_1^\infty \gamma_m^{*-1} \coth \gamma_m^* x_0 \psi_m^*(y)\,\psi_m^*(y'). \tag{6.19}$$

Equation (6.18) is a homogeneous Fredholm equation of the first kind for $U(y)$; the existence of an edge wave may be established by finding a non-trivial solution. The equation has been formulated on the basis that $d < 1$; if $d > 1$ a similar procedure may be followed. The same equation (6.18) is obtained provided that d is replaced by one.

Equation (6.18) is difficult to handle as to find an edge wave we must show that the kernel $K(y,y')$ is a singular operator (i.e. $KU = 0$ has a non-trivial solution). We therefore follow a procedure used by Miles (1967). First define the inner product

$$\langle f, g \rangle = \int_{-d}^0 f(y)\,g(y)\,dy, \tag{6.20}$$

then introduce a new kernel

$$K_s(y,y') = \sum_1^\infty \gamma_m^{-1} \psi_m(y)\,\psi_m(y') + \sum_1^\infty \gamma_m^{*-1} \coth \gamma_m^* x_0 \psi_m^*(y)\,\psi_m^*(y'). \tag{6.21}$$

Equation (6.18) becomes

$$\int_{-d}^{0} K_s(y,y')\,U(y')\,dy' = a\psi(y) - a^*\psi^*(y), \tag{6.22}$$

where $$-\gamma a = \langle U,\psi\rangle, \quad a^*\gamma^*\tanh\gamma^* x_0 = \langle U,\psi^*\rangle. \tag{6.23}$$

We now observe that for an arbitrary non-trivial function $f(y)$

$$\langle K_s f,f\rangle = \sum_1^\infty \gamma_m^{-1}\langle f,\psi_m\rangle^2 + \sum_1^\infty \gamma_m^{*-1}\coth\gamma_m^* x_0\,\langle f,\psi_m^*\rangle^2, \tag{6.24}$$

and is necessarily positive (note that the same statement cannot be made about K as $\gamma^*\tanh\gamma^* x_0$ may be negative). It follows that the operator associated with the kernel $K_s(y,y')$ has a unique inverse. Further, a proof, along analogous lines to that given by Bartholomeusz (1958) in a similar situation, may be obtained to show that the integral equations

$$\int_{-d}^{0} K_s(y,y')\,v(y')\,dy' = \psi(y)$$

and $$\int_{-d}^{0} K_s(y,y')\,v^*(y')\,dy' = \psi^*(y) \quad \text{in} \quad -d < y < 0 \tag{6.25}$$

have unique solutions. It follows from (6.22) that

$$U(y) = av(y) - a^*v^*(y), \tag{6.26}$$

and substitution into (6.23) yields the equations

$$\left.\begin{aligned} -\gamma a &= a\langle v,\psi\rangle - a^*\langle v^*,\psi\rangle, \\ a^*\gamma^*\tanh\gamma^* x_0 &= a\langle v,\psi^*\rangle - a^*\langle v^*,\psi^*\rangle. \end{aligned}\right\} \tag{6.27}$$

If there is an edge wave then these equations must have a non-trivial solution for a and a^*. The condition that there exists an edge wave is therefore

$$(\gamma + S_{11})(\gamma^*\tanh\gamma^* x_0 + S_{22}) - S_{12}S_{21} = 0, \tag{6.28}$$

where $$S_{11} = \langle v,\psi\rangle, \quad S_{22} = \langle v^*,\psi^*\rangle, \quad S_{12} = \langle v^*,\psi\rangle, \quad S_{21} = \langle v,\psi^*\rangle. \tag{6.29}$$

The symmetry of the operator K_s implies that $S_{12} = S_{21}$. Also (6.24) shows that S_{11} and S_{22} are necessarily positive, and applying (6.24) with f replaced by $v + \alpha v^*$, α arbitrary, implies that

$$S_{11}S_{22} \geqslant S_{12}^2. \tag{6.30}$$

First, let us suppose that $d \geqslant 1$. Then $k^2 \geqslant k^{*2}$, and so γ^* is real and positive, since γ must be real and positive for an edge wave. But then (6.30) shows that (6.28) cannot be satisfied and hence there are no edge waves. This result can also be established by observing that, if γ and γ^* are both real, $\langle Kf,f\rangle$ is always positive for a non-trivial f.

Next let $d < 1$, in which case $k^2 < k^{*2}$. If γ^* is real there are no edge waves; hence let

$$\gamma^* = -\,im^*, \quad m^* = (k^{*2} - l^2)^{\frac{1}{2}}. \tag{6.31}$$

Edge waves are possible only if

$$k^{*2} \geqslant l^2 > k^2, \tag{6.32}$$

or $$l\tanh l > \sigma^2 \geqslant l\tanh ld,$$

where the second inequality follows from (6.1) and (6.6). The condition (6.28) becomes

$$(\gamma + S_{11}) m^* \tan m^* x_0 = \gamma S_{22} + (S_{11} S_{22} - S_{12}^2). \tag{6.33}$$

We shall now show that for small values of l this condition can be satisfied. From (6.32) small l implies that σ is comparable with l, and hence k and k^* are also comparable with l. Indeed

$$\sigma^2 = k^2 + O(l^4) = k^{*2} d + O(l^4). \tag{6.34}$$

Let $$K_s v_0 = 1, \quad K_s v_1 = \tfrac{1}{2}(1+y)^2, \quad K_s v_1^* = \tfrac{1}{2} d^{-1}(d+y)^2. \tag{6.35}$$

On expanding (6.1), (6.3), (6.6) and (6.8) for small σ^2 it follows that

$$\left.\begin{aligned} Nv &= v_0 + \tfrac{1}{2}\sigma^2 v_1 + O(\sigma^4), \\ N^* v^* &= v_0 + \tfrac{1}{2}\sigma^2 v_1^* + O(\sigma^4). \end{aligned}\right\} \tag{6.36}$$

Also v_0, v_1 and v_1^* will be $O(1)$ as $l \to 0$. Substitution into (6.29) shows that

$$\left.\begin{aligned} S_{11} = \langle v_0, 1\rangle + O(\sigma^2), \quad S_{22} = d^{-1}\langle v_0, 1\rangle + O(\sigma^2), \\ S_{11} S_{22} - S_{12}^2 = O(\sigma^4). \end{aligned}\right\} \tag{6.37}$$

For a fixed (but small) value of l, let σ^2 increase from $l \tanh ld$ to $l \tanh l$. Then γ decreases monotonically from $l\{(1-d)^{\frac{1}{2}} + O(l^2)\}$ to zero; and hence the right-hand side of (6.33) varies continuously from $S_{22} l\{(1-d)^{\frac{1}{2}} + O(l^2)\}$ to a positive value $O(l^4)$. But m^* increases monotonically from zero to $l\{(d^{-1}-1)^{\frac{1}{2}} + O(l^2)\}$. If $m^* x_0$ remains less than $\frac{1}{2}\pi$, then the left-hand side of (6.33) varies continuously from zero to $S_{11} m^* \tan m^* x_0$, and at σ^2 equal to $l \tanh l$ this term will be positive and greater than the right-hand side of (6.33). On the other hand, if $m^* x_0$ exceeds $\frac{1}{2}\pi$, then the left-hand side of (6.32) varies continuously from zero to infinity as σ^2 increases from $l \tanh ld$. In either case there must be at least one value of σ^2 for which (6.33) is satisfied.

The values of σ^2 which satisfy (6.33) are given approximately by

$$m^* d \tan m^* x_0 + O(\gamma m^* \tan m^* x_0) = \gamma\{1 + O(\sigma^2)\} + O(\sigma^4). \tag{6.38}$$

There are two cases to consider. First suppose that x_0 is $O(1)$, and then $\tan m^* x_0$ is $O(l)$, The left-hand side of (6.38) is $O(l^2)$ and so γ is $O(l^2)$. Equation (6.38) now simplifies to

$$\left.\begin{aligned} \gamma &= x_0(1-d)\, l^2 + O(l^4), \\ \text{or} \quad \sigma^2 &= l^2 - l^4(\tfrac{1}{3} + x_0^2(1-d)^2) + O(l^6). \end{aligned}\right\} \tag{6.39}$$

This agrees with the results obtained in §4, equation (4.38). Next suppose that lx_0 is $O(1)$. Then $\tan m^* x_0$ is $O(1)$ and so γ is only $O(l)$. Equation (6.38) reduces to

$$\left.\begin{aligned} \gamma &= m^* d \tan m^* x_0 + O(l^2), \\ \text{or} \quad (l^2 - \sigma^2)^{\frac{1}{2}} &= d(\sigma^2 d^{-1} - l^2)^{\frac{1}{2}} \tan\{x_0(\sigma^2 d^{-1} - l^2)^{\frac{1}{2}}\} + O(l^2). \end{aligned}\right\} \tag{6.40}$$

Neglecting the error term, this is the result obtained by Longuet-Higgins (1967), using the shallow-water approximation. There are $n+1$ edge-wave modes, where n is the largest integer such that

$$n\pi < l x_0 (d^{-1} - 1)^{\frac{1}{2}}. \tag{6.41}$$

However, this result has been established here on the hypothesis that l is small, and although (6.41) implies that all the modes for which $n > 0$ have a low frequency cut-off, (6.41) should also be regarded as a statement about the magnitude of x_0. For any prescribed x_0, the shallow-water approximation provides information only about the $n+1$ modes which satisfy (6.41) as $l \to 0$.

REFERENCES

BALL, F. K. 1967 *Deep-Sea Res.* **14**, 79–88.

BARTHOLOMEUSZ, E. F. 1958 *Proc. Camb. Phil. Soc.* **54**, 106–118.

ECKART, C. 1951 Surface waves in water of variable depth. *Scripps Inst. of Ocean. Wave Rep.* no. 100, ref. 51–12.

HERSCH, J. & PAYNE, L. E. 1968 *Z. angew. Math. Phys.* **19**, 802–817.

LONGUET-HIGGINS, M. S. 1967 *J. Fluid Mech.* **29**, 781–821.

MEYER, R. E. 1971 Resonance of unbounded water bodies. In *Mathematical Problems in the Geophysical Sciences*: 1, pp. 189–228. *Geophysical Fluid Dynamics, Lectures in Applied Mathematics*, vol. 13. American Mathematical Society.

MILES, J. W. 1967 *J. Fluid Mech.* **28**, 755–767.

SHEN, M. C., MEYER, R. E. & KELLER, J. B. 1968 *Phys. Fluids*, **11**, 2289–2304.

URSELL, F. 1952 *Proc. Roy. Soc.* A **214**, 79–97.

WEHAUSEN, J. V. 1960 Surface waves. *Handbuch der Physik* (3), pp. 446–778. Springer.